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Vacuum polarization in muonic atoms: the Lamb shift at low and medium Z

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Abstract. In muonic atoms the Uehling potential (an effect of a free electronic vacuum polarization loop) is responsible for the leading contribution to the Lamb shift causing the splitting of states with $\Delta n = 0$ and $\Delta l \neq 0$. Here we consider the Lamb shift in the leading nonrelativistic approximation, i.e., within an approach based on a certain Schrödinger equation. That is valid for low and medium Z as long as $(Z\alpha)^2 \ll 1$. The result is a function of a few parameters, including $\kappa = Z\alpha m_\mu/m_e$, n and l. We present various asymptotics and in particular we study a region of validity of asymptotics with large and small κ. Special attention is paid to circular states, which are considered in a limit of $n \gg 1$.

1 Introduction

The Gross structure of energy levels in all kinds of hydrogen-like atoms is generally of the same form determined by the Schrödinger-Coulomb equation

$$E(nl_j) \simeq -\frac{(Z\alpha)^2 mc^2}{2n^2},$$

where m is the mass of the orbiting particle which is an electron in a conventional atom and a heavier particle in a muonic or exotic atom. However, details of the spectrum and, in particular, the structure of the energy levels with the same value of the principal quantum number n are different in different kinds of atoms. For example, in muonic atoms at low and medium Z the largest splitting between states with the same n is the one for states with $\Delta l \neq 0$ (the Lamb splitting) which is essentially a nonrelativistic effect.

In the nonrelativistic approximation the leading contribution to the Lamb shift in muonic atoms (i.e., the Uehling correction) has been known analytically for a while for certain levels [1], however, only numerical results used to be quoted in the textbooks (see, e.g., [2]). A reason for that is the complicated form of the analytic expressions. For instance, in the simplest case of the ground

state the result is of the form [1]

$$\Delta E(1s) = -\frac{\alpha}{3\pi}(Z\alpha)^2 mc^2 \left\{ -\frac{4 + \kappa^2 - 2\kappa^4}{\kappa^3} A(\kappa) \right.$$
$$\left. + \frac{4 + 3\kappa^2}{\kappa^3}\frac{\pi}{2} - \frac{12 + 11\kappa^2}{3\kappa^2} \right\}, \quad (1)$$

where

$$A(\kappa) = \frac{\arccos(\kappa)}{\sqrt{1-\kappa^2}} = \frac{\ln\left(\kappa + \sqrt{\kappa^2-1}\right)}{\sqrt{\kappa^2-1}}, \quad (2)$$

$$\kappa = \frac{Z\alpha m}{m_e}, \quad (3)$$

and m_e is the electron mass. Expressions for other states are similar, but more complicated. They involve functions $A(\kappa_n)$ with a characteristic parameter

$$\kappa_n = \frac{\kappa}{n}, \quad (4)$$

and coefficients similar to those in equation (1) depend on values of the principal and orbital quantum numbers, n and l.

The mass of the orbiting particle m in a non-conventional hydrogen-like atom is much above the electron mass m_e. We consider here the vacuum polarization effects for a hydrogen-like atoms with an orbiting particle, which in particular may be a muon ($m_\mu \simeq 207 m_e$;

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$\kappa \simeq 1.5\,Z$), a pion ($m_\pi \simeq 273\,m_e$; $\kappa \simeq 2\,Z$), an antiproton ($m_{\overline{p}} \simeq 1836\,m_e$; $\kappa \simeq 13\,Z$) etc. The relativistic effects for those atoms are quite different for various reasons, while the result in the leading nonrelativistic approximation is the same. Further we do not distinguish between various possibilities of the orbiting particles and mainly speak about a muon, but the equations could be applied to any orbiting particle.

Analytic results have been known for some time even for hydrogen-like atoms with a Dirac particle [3] and since recently for the case of a Klein-Gordon particle [4]. They are rather cumbersome, containing the hypergeometric function $_3F_2$ and far from being transparent. For instance, the relativistic result [3,4] for the nl states reads as a finite sum over basic integrals

$$
\begin{aligned}
K_{abc}(\widetilde{\kappa}_n) = {} & \frac{1}{2} \widetilde{\kappa}_n^c\, B\big(a+1/2, 1-b/2+c/2\big) \\
& \times {}_3F_2\big(c/2,\, c/2+1/2,\, 1-b/2+c/2; \\
& \qquad 1/2,\, a+3/2-b/2+c/2;\ \widetilde{\kappa}_n^2\big) \\
& - \frac{c}{2}\, \widetilde{\kappa}_n^{c+1}\, B\big(a+1/2, 3/2-b/2+c/2\big) \\
& \times {}_3F_2\big(c/2+1,\, c/2+1/2,\, 3/2-b/2+c/2; \\
& \qquad 3/2,\, a+2-b/2+c/2;\ \widetilde{\kappa}_n^2\big), \quad (5)
\end{aligned}
$$

where $_3F_2\big(\alpha_1, \alpha_2, \alpha_3;\ \beta_1, \beta_2;\ z\big)$ stands for the generalized hypergeometric function (see, e.g., [5]) and $B\big(\alpha_1, \alpha_2\big)$ is the beta function. The parameters a, b, c are linear functions of n and l in the nonrelativistic case, while for the relativistic results they contain certain additions of relativistic corrections which go to zero at the limit of $(Z\alpha) \to 0$. The argument of $_3F_2$, $\widetilde{\kappa}_n^2$, is reduced to κ_n^2 in the nonrelativistic approximation.

Meanwhile, muonic atoms offer a special region of parameters where the result can be essentially simplified (see e.g. [3,6]). For instance, the Uehling correction for the ground state [3,6] (cf. Eq. (1)) takes the form

$$
\Delta E(1s) \simeq -\frac{\alpha}{\pi}\,(Z\alpha)^2 mc^2 \left(\frac{2}{3}\ln\big(2\kappa\big) - \frac{11}{9}\right). \quad (6)
$$

The simplification is possible because in the range of medium Z we can apply for the ground state a double expansion over two parameters:

$$
\begin{aligned}
Z\alpha &\ll 1, \\
\kappa &\gg 1. \quad (7)
\end{aligned}
$$

Here and further we consider only a leading non-relativistic approximation (i.e., the leading term of the $Z\alpha$ expansion).

Highly excited states in muonic and exotic atoms are of particular interest for precision measurements because they offer a certain suppression of the interaction between the nucleus and the orbiting particle. The n dependence of theoretical expressions, even of the simplest asymptotics, is not a trivial issue. One can see from expressions with the generalized hypergeometric function $_3F_2$ that while

the argument is κ_n^2, the parameters are n dependent and in fact in actual situations some are proportional to n.

In particular, the parametrical structure of asymptotic results for high κ_n can be easily understood in the coordinate representation since the characteristic radius of the potential is the Compton wave length of an electron $\hbar/m_e c$ and the radius of atomic states is typically $\hbar n^2/Z\alpha m c$. Thus, the actual expansion is in n/κ_n, rather than just in $1/\kappa_n$. A similar situation is with low κ_n. Study of the n dependence and a determination of a real parameter of expansion are important to find the range of validity of various asymptotics.

Here we derive a general expression for the Lamb shift at medium values of the nuclear charge Z. Finally the vacuum polarization correction is presented in the leading nonrelativistic approximation in the form

$$
\Delta E(nl) = \frac{\alpha}{\pi}\,(Z\alpha)^2\,\frac{mc^2}{n^2}\,F_{nl}(\kappa_n)\,. \quad (8)
$$

The Lamb shift splits the levels with $\Delta n = 0$ and $\Delta l \neq 0$ and for this reason we also consider a specific difference

$$
\Phi_{nll'}(\kappa_n) = F_{nl}(\kappa_n) - F_{nl'}(\kappa_n)\,, \quad (9)
$$

and typically for our calculations $l' = l+1$.

We find in this paper asymptotics for low and high κ_n and determine regions of their validity. We study in more detail circular states and show that for them the low-κ_n expansion is an expansion over $n\kappa_n = \kappa$, while the high-κ_n asymptotics is actually an expansion over n^2/κ.

Additionally to well-defined regions of these expansions ($\kappa \ll 1$ or $n^2/\kappa \ll 1$) there are also two intermediate regions:

- low κ_n, when $\kappa_n \ll 1$, but $n\kappa_n \sim 1$;
- high κ_n, when $\kappa_n \gg 1$, but $\kappa_n/n \sim 1$.

We discuss behavior of the Uehling correction in these two specific regions.

2 The Uehling correction in the nonrelativistic approximation: general consideration for the Lamb shift

Let us first remind how the Uehling correction is calculated in a general case. The Lamb shift in muonic atoms is a result of perturbing the Coulomb potential

$$
V_C(r) = -\frac{Z\alpha}{r} \quad (10)
$$

by the Uehling potential [7]

$$
V_U(r) = \frac{\alpha}{\pi} \int_0^1 dv\, \frac{v^2(1-v^2/3)}{1-v^2} \left(-\frac{Z\alpha}{r} e^{-\lambda r}\right)\,, \quad (11)
$$

where the dispersion 'photon' mass

$$
\lambda = \frac{2m_e}{\sqrt{1-v^2}} \quad (12)
$$

plays a role of the inverse Yukawa radius. Here and for other calculations in this paper we use relativistic units in which $\hbar = c = 1$, while for final results we restore c and $\hbar$ if necessary.

The Lamb shift in the nonrelativistic approximation is of the form

$$\Delta E^{(0)}(nl) = \int dr\, r^2 |R_{nl}|^2 V_U(r)$$
$$= \frac{\alpha}{\pi} (Z\alpha)^2 \frac{m}{n^2} F_{nl}(\kappa_n), \qquad (13)$$

where $R_{nl}(r)$ is the radial part of the Schrödinger wave function in a hydrogen-like atom

$$\varphi_{nlm}(\mathbf{r}) = R_{nl}(r) Y_{lm}(\mathbf{r}/r) . \qquad (14)$$

Applying the well-known analytic expression for $R_{nl}(r)$ to equation (13) and integrating over r, we obtain (see Eq. (f.9) in [8])

$$\Delta E^{(0)}(nl) = -\frac{\alpha(Z\alpha)}{2n\,\pi} \left(\frac{2Z\alpha m}{n} \right)^{2l+3} \frac{(n+l)!}{(2l+1)!(n-l-1)!}$$
$$\times \int_0^1 dv\, \frac{v^2(1-v^2/3)}{1-v^2} \lambda^{2(n-l-1)} \left(\frac{2Z\alpha m}{n} + \lambda \right)^{-2n}$$
$$\times {}_2F_1 \left(-n+l+1, -n+l+1; 2l+2; \left(\frac{2Z\alpha m}{n\lambda} \right)^2 \right) . \qquad (15)$$

After replacing the hypergeometric function by an explicit finite sum, we integrate over v and arrive at the following expression for F_{nl}:

$$F_{nl}(\kappa_n) = -\frac{(n+l)!}{(n-l-1)!} \sum_{i=0}^{n-l-1} \frac{1}{(2l+i+1)!} \frac{1}{i!}$$
$$\times \left(\frac{(n-l-1)!}{(n-l-i-1)!} \right)^2 \frac{1}{\kappa_n^{2(n-l-1-i)}}$$
$$\times \left[K_{1,2(n-l-i),2n}(\kappa_n) - \frac{1}{3} K_{2,2(n-l-i),2n}(\kappa_n) \right], \qquad (16)$$

where the integrals

$$K_{abc}(\kappa) = \int_0^1 dv\, \frac{v^{2a}}{(1-v^2)^{b/2}} \left(\frac{\kappa\sqrt{1-v^2}}{1+\kappa\sqrt{1-v^2}} \right)^c \qquad (17)$$

can be expressed in general in terms of the generalized hypergeometric functions (5) [3]. Here we mainly follow our notation in [3], but the definition of the integral K (see also [4]) is different from the related integral I there. While in the nonrelativistic limit, when $\epsilon = 0$ and the parameter c is integer, $K_{a,b,c}(\kappa) = I_{a,b,c}(\kappa)$, in the relativistic case with non-integer c the notation is $K_{a,b,c}(\kappa) = I_{a,b,c+2\epsilon}(\kappa, \epsilon)$.

We note that for integer a, b, c the result can be expressed in terms of elementary functions. Using recursive relations (cf. [3])

$$\frac{1}{\kappa^{c+1}} K_{a,b,c+1}(\kappa) = -\frac{1}{c} \frac{\partial}{\partial\kappa} \left[\frac{1}{\kappa^c} K_{a,b,c}(\kappa) \right], \qquad (18)$$

$$K_{a,b+1,c+1}(\kappa) = \frac{\kappa^2}{c} \frac{\partial}{\partial\kappa} K_{a,b,c}(\kappa) \qquad (19)$$

we express the correction for an arbitrary state through the expression for the ground state

$$F_{nl}(\kappa_n) = \frac{(n+l)!}{(n-l-1)!(2n-1)!} \sum_{i=0}^{n-l-1} \frac{1}{(2l+i+1)!} \frac{1}{i!}$$
$$\times \left(\frac{(n-l-1)!}{(n-l-i-1)!} \right)^2 \left(\frac{1}{\kappa_n} \right)^{2(n-l-1-i)}$$
$$\times \left(\kappa_n^2 \frac{\partial}{\partial\kappa_n} \right)^{2(n-l-i-1)} \kappa_n^{2(l+i+1)} \left(\frac{\partial}{\partial\kappa_n} \right)^{2(l+i)} \frac{F_{10}(\kappa_n)}{\kappa_n^2} . \qquad (20)$$

The result for F_{10}

$$F_{10}(\kappa) = -K_{122}(\kappa) + \frac{1}{3} K_{222}(\kappa), \qquad (21)$$

which follows from equation (16), is known in simpler terms and in particular in terms of elementary functions (see Eq. (1)). The general expression (20) now presents a correction for any states in terms of elementary functions. Such an expression is also very useful to derive various asymptotics once we find related asymptotics for $F_{10}(\kappa)$. Another way of the F_{nl} presentation as a single finite sum can be found in [9].

3 Asymptotic behavior at large κ_n

In the case of $\kappa_n \gg 1$ we can use asymptotics for the ground state function F_{10} (cf. [3] and Eq. (6))

$$F_{10}(\kappa) = -\left[\frac{2}{3} \ln(2\kappa) - \frac{11}{9} \right] - \frac{\pi}{2} \frac{1}{\kappa} + \frac{3}{2} \frac{1}{\kappa^2}$$
$$- \frac{2\pi}{3} \frac{1}{\kappa^3} + \left[\frac{5}{4} \ln(2\kappa) + \frac{1}{16} \right] \frac{1}{\kappa^4}$$
$$+ \left[\frac{7}{12} \ln(2\kappa) - \frac{5}{18} \right] \frac{1}{\kappa^6} + \dots \qquad (22)$$

An expression for an arbitrary state can be also derived as an expansion over $1/\kappa_n$. Here we present few first terms (cf. [3,9])

$$F_{nl}(\kappa_n) = -\frac{2}{3} \left[\ln(2\kappa_n) + \psi(1) - \psi(n+l+1) - \frac{5}{6} \right]$$
$$- \frac{\pi}{2} \frac{n}{\kappa_n} + \frac{1}{2} \left[n(2n+1) + (n+l)(n-l-1) \right] \frac{1}{\kappa_n^2}$$
$$- \frac{\pi}{9} \left[(2n+1)(n+1) + 3(n+l)(n-l-1) \right] \frac{n}{\kappa_n^3} + \dots , \qquad (23)$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function and

$$\psi(n) - \psi(1) = \sum_{i=1}^{n-1} \frac{1}{i}.$$

The results for the asymptotics of the difference (9) related to the Lamb shift are much simpler than the result for each level separately:

$$\Phi_{n,l-1,l}(\kappa_n) = -\frac{2}{3}\frac{1}{n+l} + \frac{l}{\kappa_n^2} - \frac{2\pi}{3}\frac{n\,l}{\kappa_n^3} + \dots \quad (24)$$

To test our calculations, we consider a limit $\ln \kappa_n \gg 1$ and find the leading logarithmic term within the effective charge approach with the help of a substitution

$$Z\alpha \longrightarrow Z\alpha(\kappa_n) = Z\alpha\left(1 + \frac{2\alpha}{3\pi}\ln\kappa_n\right). \quad (25)$$

The result reads

$$F_{nl}^{\log}(\kappa_n) = -\frac{2}{3}\ln\kappa_n. \quad (26)$$

The logarithmic contribution vanishes for the Lamb splitting $\Phi_{n,l-1,l}$. The logarithmic results are in agreement with the direct calculations above.

4 Asymptotic behavior at large κ_n and large n

We note that the asymptotic coefficients depend on n and one may wonder about their behavior at high n. To study this we apply the well-known expansion for $\psi(z)$ at high z

$$\psi(z+1) = \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \dots$$

The result for the Uehling correction reads

$$
\begin{aligned}
F_{nl}(\kappa_n) = &-\frac{2}{3}\left[\ln\left(\frac{2\kappa_n}{n+l}\right) - \mathcal{C} - \frac{5}{6}\right.\\
&\left. - \frac{1}{2}\frac{1}{(n+l)} + \frac{1}{12}\frac{1}{(n+l)^2} + \dots\right]\\
&- \frac{\pi}{2}\left(\frac{n}{\kappa_n}\right) + \frac{3n^2 - l(l+1)}{2n^2}\left(\frac{n}{\kappa_n}\right)^2\\
&- \frac{\pi}{9}\frac{5n^2 - 3l(l+1) - 1}{n^2}\left(\frac{n}{\kappa_n}\right)^3 + \dots, \quad (27)
\end{aligned}
$$

where $\mathcal{C} = -\psi(1) = 0.577\,215\,665...$ is Euler's constant. Certain simplifications are achieved once we do an assumption on a particular relation between values of l and n.

4.1 Low-l states

An important feature of the result in equation (27) is that the parameter of expansion is rather n/κ_n than $1/\kappa_n$. For

instance, our explicit result for F_{nl} at $n \gg 1$ and low l ($l \ll n$) is

$$
\begin{aligned}
F_{nl}(\kappa_n) = &-\frac{2}{3}\left[\ln\left(\frac{2\kappa_n}{n}\right) - \mathcal{C} - \frac{5}{6} - \frac{2l+1}{2n}\right.\\
&\left. + \frac{6l(l+1)+1}{12n^2} + \dots\right] - \frac{\pi}{2}\left(\frac{n}{\kappa_n}\right)\\
&+ \left[\frac{3}{2} - \frac{l(l+1)}{2n^2}\right]\left(\frac{n}{\kappa_n}\right)^2\\
&- \pi\left[\frac{5}{9} - \frac{3l(l+1)-1}{9n^2}\right]\left(\frac{n}{\kappa_n}\right)^3 + \dots \quad (28)
\end{aligned}
$$

We keep here the l dependence in the $1/n^2$ terms in order to derive a related result for the Lamb splitting

$$
\begin{aligned}
\Phi_{n,l-1,l}(\kappa_n) = \frac{1}{n}\left\{-\frac{2}{3} + \frac{2l}{3n}\right.\\
\left. + \frac{1}{n}\left(\frac{n}{\kappa_n}\right)^2 - \frac{2\pi}{3n}\left(\frac{n}{\kappa_n}\right)^3 + \dots\right\}. \quad (29)
\end{aligned}
$$

We note that the expansion in (28) and (29) is effectively done in n/κ_n. Meanwhile, the leading term in (29) is suppressed by a factor of $1/n$ and the two first corrections are additionally suppressed by $1/n$.

4.2 Near circular states

After studying $n \gg 1$ at low l, we turn to another case of $n \gg 1$ at low values of the radial quantum number $n_r = n - l - 1 \sim 1$. In particular, $n_r = 0$ is related to the so-called circular state. In the limit of high κ_n and n we obtain

$$
\begin{aligned}
F_{n,n-n_r-1}(\kappa_n) = &-\frac{2}{3}\left[\ln\left(\frac{\kappa_n}{n}\right) - \mathcal{C} - \frac{5}{6} + \frac{2n_r+1}{4n} + \dots\right]\\
&- \frac{\pi}{2}\left(\frac{n}{\kappa_n}\right) + \left[1 + \frac{2n_r+1}{2n} + \dots\right]\left(\frac{n}{\kappa_n}\right)^2\\
&- \frac{\pi}{9}\left[2 + \frac{6n_r+3}{n} + \dots\right]\left(\frac{n}{\kappa_n}\right)^3 + \mathcal{O}\left(\left(\frac{n}{\kappa_n}\right)^4\right). \quad (30)
\end{aligned}
$$

In the same limit the specific difference related to the Lamb shift is

$$
\begin{aligned}
\Phi_{n,n-n_r-2,n-n_r-1}(\kappa_n) = \\
\frac{1}{n}\left\{-\frac{1}{3} + \left(\frac{n}{\kappa_n}\right)^2 - 2\pi\left(\frac{n}{\kappa_n}\right)^3 + \dots\right\}. \quad (31)
\end{aligned}
$$

The difference is suppressed by $1/n$, as well as for low l, but, in contrast to equation (29), there is no additional suppression. As a result, we see that the high-κ_n expansions above (cf. [3,9]) are valid only in the case of $\kappa_n \gg n$, which reduces the range of their applicability drastically. We consider the case of $\kappa_n \gg 1$, but not $\kappa_n \gg n$ in Section 6.

5 Asymptotics at low κ

In principle we are interested in high rather than in low κ values, because the problem is related to muonic and exotic atoms. However, for high n, even for $\kappa \gg 1$ we can easily arrive at a situation when $\kappa_n = \kappa/n \ll 1$ and thus this region is of interest.

The asymptotic behavior of $F_{nl}(\kappa_n)$ at small values of κ_n was studied in [3] (see also [9]). Various approaches can be used for that. One may start from our expression (20) (cf. [3,9]), taking into account that

$$F_{10}(\kappa) = -\frac{4\kappa^2}{15} + \frac{5\pi\kappa^3}{48} - \frac{12\kappa^4}{35} + \frac{7\pi\kappa^5}{64} - \frac{64\kappa^6}{189}$$
$$+ \frac{27\pi\kappa^7}{256} - \frac{32\kappa^8}{99} + \frac{77\pi\kappa^9}{768} - \frac{1536\kappa^{10}}{5005} + \dots, \tag{32}$$

or apply equation (16) with K_{abc} presented in terms of integral (17) or of generalized hypergeometric functions (5). Actually, the latter is the most straightforward way to obtain a low-κ_n expansion. In case $\kappa_n \ll 1$ the expansion reads

$$F_{nl}(\kappa_n) = -\frac{(n+l)!\,\kappa_n^{2l+2}}{(2l+1)!(n-l-1)!}\left\{\frac{1}{2(l+1)}\frac{(2l+4)!!}{(2l+5)!!}\right.$$
$$-\ \pi(n\,\kappa_n)\frac{1}{2l+3}\frac{(2l+5)!!}{(2l+6)!!}$$
$$+\ (n\,\kappa_n)^2\left(\frac{4l+5}{l+1}+\frac{l+1}{n^2}\right)\frac{1}{4(l+2)}\frac{(2l+6)!!}{(2l+7)!!}$$
$$-\ \pi(n\,\kappa_n)^3\left(\frac{4l+7}{l+1}+\frac{3l+5}{n^2}\right)\frac{1}{6(2l+5)}\frac{(2l+7)!!}{(2l+8)!!}$$
$$+\ \mathcal{O}((n\kappa_n)^4)\left.\right\}. \tag{33}$$

The first term of this expansion is obtained in [9] and is in agreement with our expression. As one can see, the series is in fact over $n\kappa_n = \kappa$ rather than κ_n. That sets a condition for applicability of the low-κ_n asymptotics as $\kappa_n \ll 1/n$. In particular, it means that the asymptotics equation (33) cannot be applied for Rydberg states even for the muonic hydrogen, i.e., for the smallest possible Z ($Z = 1$), where $\kappa \sim 1.5$ and $\kappa_n \sim 1.5/n$.

6 High n asymptotic behavior

We see that while we expand the generalized hypergeometric function in terms of either κ_n or $1/\kappa_n$, the real parameters of both expansions involve a factor of n directly. That is due to the increase of the coefficients of the κ_n- or $1/\kappa_n$-expansions with n which technically originates from the expansion of the factor

$$\left(\frac{\kappa_n\sqrt{1-v^2}}{1+\kappa_n\sqrt{1-v^2}}\right)^c \tag{34}$$

in the basic integral $K_{abc}(\kappa_n)$, while $c = 2n$.

We note that a consideration of high n is not unrealistic. For instance, in [neutral] antiprotonic helium for realistic levels [10] we find $Z = 2$, $n \simeq 30 \gg 1$, $\kappa \simeq 27 \gg 1$, $\kappa_n \simeq 1$. One of the reasons to study high-n states is that they very weakly interact with the nucleus, especially if a value of l is also high. Such an immunity to the nuclear-structure effects is an advantage from both theoretical and experimental point of view. Therefore and also because of simplifications in calculations we consider below circular or near circular states at $n \gg 1$.

6.1 Limit of low κ_n for the near-circular states

The combination of the K_{abc} integrals which actually enters the equation for the vacuum-polarization energy shifts is

$$K_{bc}(\kappa_n) = K_{1bc}(\kappa_n) - \frac{1}{3}K_{2bc}(\kappa_n). \tag{35}$$

We find that $b \ll n$ for the near-circular states, and, as long as we use (16), $c = 2n$ for any state.

Once we know the general expression (5) in terms of $_3F_2$ (cf. [3]), we can consider in each order of the κ_n expansion only terms leading in n (we did above a similar procedure to prove for the few first terms of series that the expansion is over $n\kappa_n$, and not over just κ_n).

Collecting the leading in n terms we arrive at the result in the limit $n \gg 1$, $\kappa_n \ll 1$ and $b \ll n$

$$K_{b,2n}(\kappa_n) \simeq \frac{(\kappa_n)^{2n}}{2}B\left(n,\frac{3}{2}\right)\left[\,_2F_1\left(n,n;\frac{1}{2};\kappa_n^2\right)\right.$$
$$\left. -2(n\kappa_n)\,_2F_1\left(n,n;\frac{3}{2};\kappa_n^2\right)\right]. \tag{36}$$

We note that

$$_2F_1(n,n;\nu;\kappa_n^2) \simeq \Gamma(\nu)(n\kappa_n)^{1-\nu}I_{\nu-1}(2n\kappa_n),$$
$$B(n,\nu) \simeq \Gamma(\nu)n^{-\nu} \tag{37}$$

at $\nu \ll n$, where $I_\nu(z)$ is the modified Bessel function. For the latter one can apply the well-known explicit expressions for $\nu = 1/2, 3/2$ and we arrive at the expression

$$K_{b,2n}(\kappa_n) \simeq \frac{\sqrt{\pi}\kappa_n^{2n}}{4n^{3/2}}e^{-2n\kappa_n}. \tag{38}$$

To express the correction to energy $F_{nl}(\kappa_n)$ in terms of the basic integrals $K_{b,2n}(\kappa_n)$ for near-circular states ($n_r = n - l - 1 \ll n$) we need to transform the related coefficients in (16) in the limit of high n. We note that the integral $K_{b,2n}(\kappa_n)$ does not depend on b in the leading $1/n$-approximation and thus the l dependence of the correction comes from the l dependence of the coefficients of (16). Eventually we find

$$F_{nl}(\kappa_n) \simeq -\frac{\sqrt{\pi}\,\kappa_n^{2l+2}}{4n^{3/2}}\frac{(2n)^{n_r}}{n_r!}e^{-2n\kappa_n}$$
$$= -\frac{\sqrt{\pi}}{4n^{3/2}}\frac{\kappa_n^{2n}e^{-2n\kappa_n}}{n_r!}\left(\frac{2n}{\kappa_n}\right)^{n_r}. \tag{39}$$

To conclude this consideration we need to discuss the accuracy and validity of our derivation. It is valid for $\kappa_n \ll 1$ and $\kappa \sim 1$ and the corrections are of relative order $1/n$. In the case of $\kappa \ll 1$ it is consistent with the leading term of the low-κ_n expansion (33).

6.2 The limit of high κ_n for the near-circular states

For $\kappa_n \gg 1$ we also consider only near-circular states, for which $b \sim n_r = n - l - 1 \ll n$. We can rewrite $K_{b,2n}(\kappa_n)$ in terms of the basic integrals as follows

$$
\begin{aligned}
K_{b,2n}(\kappa_n) &= \int_0^1 dv\, \frac{v^2(1-v^2/3)}{(1-v^2)^{b/2}} \left(\frac{\kappa_n \sqrt{1-v^2}}{1+\kappa_n\sqrt{1-v^2}} \right)^{2n} \\
&= \int_0^1 dv\, \frac{v^2(1-v^2/3)}{(1-v^2)^{b/2}} \exp\left\{ 2n \ln\left(1 - \frac{1}{1+\kappa_n\sqrt{1-v^2}} \right) \right\}.
\end{aligned}
$$

If $b \ll n$, we can expand the exponential in the integrand and find

$$
K_{b,2n}(\kappa_n) \simeq \int_0^1 dv\, \frac{v^2(1-v^2/3)}{(1-v^2)^{b/2}} e^{-\frac{2n}{\kappa_n\sqrt{1-v^2}}}, \qquad (40)
$$

that depends upon combination of parameters κ_n/n only. After a substitute of the variable $t = 1/\sqrt{1-v^2}$ in this integral we arrive at the result

$$
K_{b,2n}(\kappa_n) \simeq \int_1^\infty dt\, \frac{\sqrt{t^2-1}}{t^{6-b}} \frac{2t^2+1}{3} e^{-\frac{2nt}{\kappa_n}}. \qquad (41)
$$

Substituting the expression into the sum in equation (16), and neglecting $n - l$ as compared with n in coefficients of the sum, we obtain

$$
\begin{aligned}
F_{nl}(\kappa_n) \simeq &-\int_1^\infty dt\, \frac{\sqrt{t^2-1}}{t^4} \frac{2t^2+1}{3} e^{-\frac{2nt}{\kappa_n}} \\
&\times \sum_{j=0}^{n_r} \frac{(n_r)!}{(n_r-j)!} \frac{1}{(j!)^2} \left(\frac{\sqrt{2n}}{\kappa_n} t \right)^{2j}. \qquad (42)
\end{aligned}
$$

Similarly to the previous subsection, we find that our derivation is appropriate for $1/\kappa_n \ll 1$ and $n/\kappa_n \sim 1$ and the result has a relative uncertainty on the order of $1/n$.

6.3 Comparison of low-κ and high-κ asymptotics

The region of highest interest is the one for high-κ is when $1/\kappa_n \ll 1$, but not $n/\kappa_n \ll 1$, since for the opposite situation ($n/\kappa_n \gg 1$) we have already known the proper asymptotic form of the correction. We note that for the region of interest κ_n/n can be about unity or even larger (e.g., as in the case of $1/\kappa_n \sim n^{-1/2} \ll 1$ and $n/\kappa_n \sim n^{1/2} \gg 1$). In particular, if $\kappa_n/n \gg 1$ the result of the t-integration in equation (42) will mainly come from a narrow region $(t-1) \ll 1$. That means that we can improve the final result once we consider a complete series
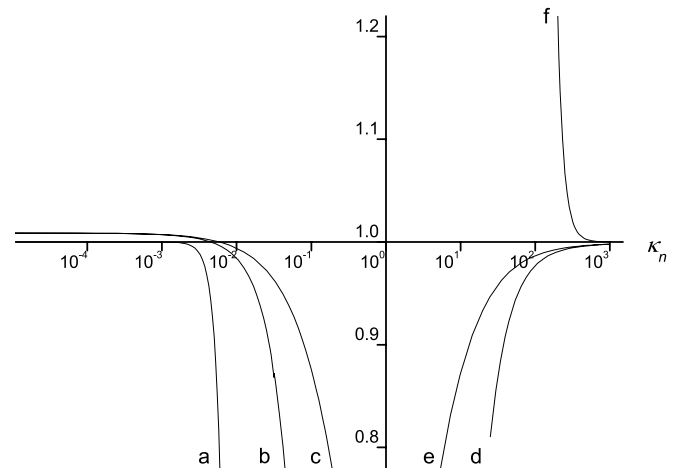

Fig. 1. Ratio of different $F_{n,n-1}$ asymptotics and its exact values for the circular state with $n = 100$: (a) – $n\kappa_n \ll 1$ (Eq. (33)), (b) – $\kappa_n \ll 1$ (Eq. (38)), (c) – $\kappa_n \ll 1$ (Eq. (44)), (d) – $\kappa_n \gg 1$ (Eq. (41)), (e) – $\kappa_n \gg 1$ (Eq. (43)), (f) – $\kappa_n/n \gg 1$ (Eq. (30)); the horizontal axis is related to $F_{n,n-1}$.

for the logarithm in Section 6.2, and setting $t = 1$ for all terms except of the leading term of the expansion

$$
\begin{aligned}
\exp\left\{ 2n \log\left(1 - \frac{1}{1+\kappa_n\sqrt{1-v^2}} \right) \right\} &\simeq \\
&\left(\frac{\kappa_n}{1+\kappa_n} \right)^{2n} e^{-\frac{2n}{\kappa_n}(t-1)}.
\end{aligned}
$$

The final estimation reads for high κ_n

$$
K_{b,2n}(\kappa_n) \simeq \left(\frac{\kappa_n}{1+\kappa_n} \right)^{2n} \int_1^\infty dt\, \frac{\sqrt{t^2-1}}{t^{6-b}} \frac{2t^2+1}{3} e^{-\frac{2nt}{\kappa_n}}. \qquad (43)
$$

We can also rewrite the result for low κ_n (Eq.(38)) as

$$
K_{b,2n}(\kappa_n) \simeq \left(\frac{\kappa_n}{1+\kappa_n} \right)^{2n} \frac{\sqrt{\pi}}{4n^{3/2}}. \qquad (44)
$$

Comparing those two asymptotics for the circular-state correction, we find that the integral $K_{2,2n}(\kappa_n)$ can be presented as a product of a factor

$$
\left(\frac{\kappa_n}{1+\kappa_n} \right)^{2n} \qquad (45)
$$

and a smooth function. The factor is varying in an extremely broad region of κ_n: from low κ_n ($\kappa_n \ll 1/n$) to large κ_n ($\kappa_n \gg n$), while the smooth function changes from being proportional to $n^{-3/2}$ to $\ln(\kappa/n^2)$. Because of so smooth behavior we expect that the asymptotics with the explicit factor equation (45) can be successfully applied for a somewhat larger region, however, their accuracy there is unclear. Various asymptotics are compared to the exact result for $n = 100$ in Figure 1. In particular we see that an explicit presentation of the factor of $(\kappa_n/(1+\kappa_n))^{2n}$ really improves agreement between the asymptotics and the exact solution.

Table 1. Asymptotics at $x \gg 1$ for the Uehling correction for the lowest s states. The correction is presented in terms of a dimensionless function F_{nl}: $\Delta E(nl) = (\alpha/\pi)\left((Z\alpha)^2 mc^2/n^2\right)F_{nl}(\kappa_n)$.

n	$F_{n0}(x)$
1	$-\dfrac{2}{3}\ln(2x)+\dfrac{11}{9}-\dfrac{\pi}{2x}+\dfrac{3}{2x^2}-\dfrac{2\pi}{3x^3}+\dots$
2	$-\dfrac{2}{3}\ln(2x)+\dfrac{14}{9}-\dfrac{\pi}{x}+\dfrac{6}{x^2}-\dfrac{14\pi}{3x^3}+\dots$
3	$-\dfrac{2}{3}\ln(2x)+\dfrac{16}{9}-\dfrac{3\pi}{2x}+\dfrac{27}{2x^2}-\dfrac{46\pi}{3x^3}+\dots$
4	$-\dfrac{2}{3}\ln(2x)+\dfrac{35}{18}-\dfrac{2\pi}{x}+\dfrac{24}{x^2}-\dfrac{36\pi}{x^3}+\dots$
5	$-\dfrac{2}{3}\ln(2x)+\dfrac{187}{90}-\dfrac{5\pi}{2x}+\dfrac{75}{2x^2}-\dfrac{70\pi}{x^3}+\dots$

Table 2. Asymptotics of the Lamb-shift-induced difference at $x \gg 1$ for the lowest states presented in terms of a dimensionless function Φ_{nab}: $\Delta E(n,l) - \Delta E(n,l') = (\alpha/\pi)\left((Z\alpha)^2 mc^2/n^2\right)\Phi_{nll'}(\kappa_n)$.

n	$\Phi_{n01}(x)$	$\Phi_{n12}(x)$	$\Phi_{n23}(x)$	$\Phi_{n34}(x)$
2	$-\dfrac{2}{9}+\dfrac{1}{x^2}$ $-\dfrac{4\pi}{3x^3}+\dots$			
3	$-\dfrac{1}{6}+\dfrac{1}{x^2}$ $-\dfrac{2\pi}{x^3}+\dots$	$-\dfrac{2}{15}+\dfrac{2}{x^2}$ $-\dfrac{4\pi}{x^3}+\dots$		
4	$-\dfrac{2}{15}+\dfrac{1}{x^2}$ $-\dfrac{8\pi}{3x^3}+\dots$	$-\dfrac{1}{9}+\dfrac{2}{x^2}$ $-\dfrac{16\pi}{3x^3}+\dots$	$-\dfrac{2}{21}+\dfrac{3}{x^2}$ $-\dfrac{8\pi}{x^3}+\dots$	
5	$-\dfrac{1}{9}+\dfrac{1}{x^2}$ $-\dfrac{10\pi}{3x^3}+\dots$	$-\dfrac{2}{21}+\dfrac{2}{x^2}$ $-\dfrac{20\pi}{3x^3}+\dots$	$-\dfrac{1}{12}+\dfrac{3}{x^2}$ $-\dfrac{10\pi}{x^3}+\dots$	$-\dfrac{2}{27}+\dfrac{4}{x^2}$ $-\dfrac{40\pi}{3x^3}+\dots$

7 Other states

Above we obtained the high-n asymptotic expressions in two specific regions of parameters where $\kappa_n \sim 1/n$ and $\kappa_n \sim n$ for the circular and the near-circular states only. We are also interested in finding asymptotics in these regions that are valid for low l.

For low κ_n and $l \ll n$ we can use an approximate relation

$$K_{2(n-l-i),2n}(\kappa_n) \simeq \kappa_n^{2n} \int_0^1 dv\, v^2 \left(1-\frac{v^2}{3}\right)\left(1-v^2\right)^{l+i}$$
$$\times \exp\left\{-2n\kappa_n\sqrt{1-v^2}\right\}, \quad (46)$$

and, neglecting l as compared with n in coefficients of the sum equation (16), obtain an approximation

$$F_{nl}(\kappa_n) \simeq -\frac{(n\kappa_n)^{2(l+1)}}{n} \int_0^1 dv\, v^2 \left(1-\frac{v^2}{3}\right)\left(1-v^2\right)^l$$
$$\times \exp\left\{-2n\kappa_n\sqrt{1-v^2}\right\} \sum_{i=0}^{n-l-1} \frac{\left(n\kappa_n\sqrt{1-v^2}\right)^{2i}}{i!\,(2l+i+1)!}.$$

We can see that the asymptotic depends upon a combination of parameters $n\kappa_n$, confirming the above-mentioned fact that it is the real parameter of expansion at low κ_n.

In the other region corresponding to $\kappa_n \sim n$ we do not see a simple way to find a proper asymptotic form for low-l states.

8 Summary

Concluding, we have to briefly discuss corrections to the results derived. Since the parameter m/M (here M is the mass of the nucleus) in muonic, pionic and other exotic atoms is not as small as in conventional atoms, an important question is the accuracy of our results obtained in the external field approximation, i.e. in the limit $m/M = 0$. In antiprotonic atoms such effects are even more important
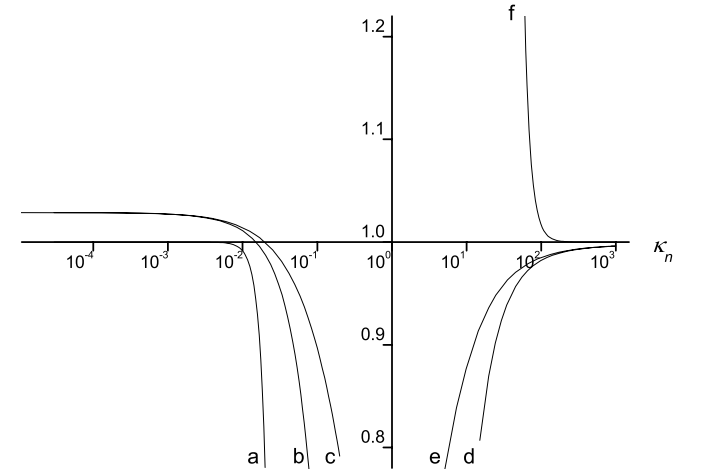

Fig. 2. Ratio of different $F_{n,n-1}$ asymptotics and its exact values for the circular state with $n = 30$: (a) $- n\kappa_n \ll 1$ (Eq. (33)), (b) $- \kappa_n \ll 1$ (Eq. (38)), (c) $- \kappa_n \ll 1$ (Eq. (44)), (d) $- \kappa_n \gg 1$ (Eq. (41)), (e) $- \kappa_n \gg 1$ (Eq. (43)), (f) $- \kappa_n/n \gg 1$ (Eq. (30)); the horizontal axis is related to $F_{n,n-1}$.

than in the muonic case. The higher m/M corrections, which are quite important for muonic atoms, can be easily taken into account for the Lamb shift in the nonrelativistic approximation by substituting the mass of the orbiting particle m for the reduced mass $m_R = mM/(m + M)$ in equation (8) and κ for $\kappa_R = Z\alpha m_R/m_e$.

The functions F_{nl} are presented above for arbitrary nl in a closed analytic form in various ways. Certain asymptotics are also presented. The results at $\kappa_n \gg 1$ for the lowest states are summarized in Table 1. They are simple and transparent. The Lamb shift result in equation (16) is obtained in the nonrelativistic approximation and is valid

for any hydrogen-like atom as far as the relativistic corrections can be neglected. Some asymptotic results for the splitting of levels with $\Delta l = 1$ at $\kappa_n \gg n$ for some low lying states are summarized in Table 2.

We studied the applicability of naive low-κ and high-κ asymptotics and found that the region where they are valid strongly depends on n. For high n we considered some additional asymptotics (see, e.g., Fig. 2 where the results are presented for a realistic value of $n = 30$). We found a sum of the leading terms for $n \gg 1$ expansions for both low-κ and high-κ cases. In particular, we found that most of the change by orders of magnitude of the Uehling correction in the circular states can be presented in terms of a simple factor $(\kappa_n/(1 + \kappa_n))^{2n}$ which is multiplied by a smooth function.

All results are obtained in the leading nonrelativistic approach and corrections due to that are of relative order $(Z\alpha)^2$. The relativistic effects will be considered elsewhere.

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